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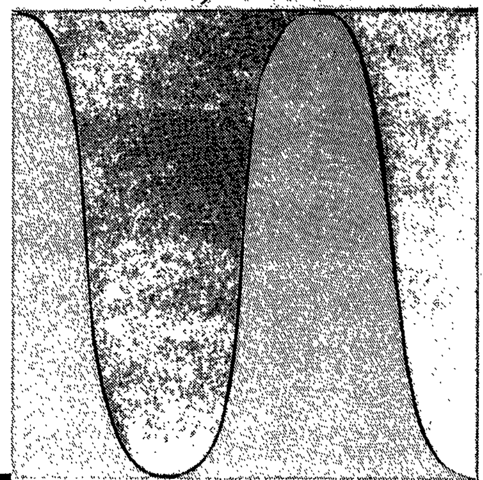

SOLUTION OF AN INTEGRAL EQUATION OF THE THIRD KIND BY SUCCESSIVE APPROXIMATIONS

P. M. Anselone, H. F. Bueckner
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MRC Technical Summary Report #345
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ABSTRACT

The integral equation

$$t\,P(t) = \mu \int_0^t (t-s)^{-\lambda}\, P(s)\, ds, \quad 0 \leqq t < \infty \;,$$

where $\mu > 0$ and $0 < \lambda < 1$, has a unique solution $P(t)$ which satisfies appropriate auxiliary conditions. Successive approximations to $P(t)$ are derived by means of a trapezoidal numerical integration scheme. They converge uniformly to $P(t)$ for $0 \leqq t < \infty$. Other approximation methods are described and numerical examples are given. Applicability of the methods to more general integral equations is indicated.

SOLUTION OF AN INTEGRAL EQUATION OF THE THIRD KIND
BY SUCCESSIVE APPROXIMATIONS

by

P. M. Anselone, H. F. Bueckner and D. Greenspan

1. Introduction. Consider the homogeneous Volterra integral equation

$$tP(t) = \mu \int_0^t (t-s)^{-\lambda} P(s)\,ds, \quad 0 \leqq t < \infty \ , \tag{1.1}$$

where $\mu > 0$, $0 < \lambda < 1$, and $P(t)$ is a bounded, continuous, non-negative, integrable function which satisfies the normalization condition

$$\int_0^\infty P(t)\,dt = 1 \ . \tag{1.2}$$

In Fredholm's classification, (1.1) is an integral equation of the third kind. Moreover, the kernel has a weak singularity. It will be shown that (1.1) has a unique solution $P(t)$ with the prescribed properties.

This paper is directed primarily to the presentation of a method for constructing successive approximations to $P(t)$ and to proving that they converge uniformly to $P(t)$. Several related approximation methods are indicated briefly. The methods are also applicable to more general integral equations of the form

$$tP(t) = \int_0^t K(t-s)\,P(s)\,ds, \qquad 0 \leqq t < \infty \ . \tag{1.3}$$

However, treatment of such generalizations is deferred to another occasion.

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Our interest in equation (1.1) was motivated by an application [1] where $P(t)$ is a probability density function. This interpretation of $P(t)$ is reflected in the auxiliary conditions, which are not independent. The following lemmas will help to clarify the situation.

Lemma 1.1. Let $P(t)$ be a continuous solution of (1.1). Then $P(t)$ is bounded and attains its maximum at some point $t \leqq (\frac{\mu}{1-\lambda})^{1/\lambda}$.

Proof. By (1.1),

$$t|P(t)| \leqq \mu\left[\int_0^t (t-s)^{-\lambda}\, ds\right]\left[\max_{0\leqq s\leqq t} |P(s)|\right],$$

$$|P(t)| \leqq \frac{\mu t^{-\lambda}}{1-\lambda} \max_{0\leqq s\leqq t} |P(s)|, \qquad t>0 . \tag{1.4}$$

Since $\frac{\mu t^{-\lambda}}{1-\lambda} < 1$ for all $t > (\frac{\mu}{1-\lambda})^{1/\lambda}$, the lemma follows.

Lemma 1.2. Let $P(t)$ be a bounded solution of (1.1). Then $P(t)$ is integrable on $[0,\infty)$ and

$$P(t) = O(t^{-1-\lambda+\epsilon}), \qquad t>0, \qquad 0<\epsilon\leqq 1 . \tag{1.5}$$

Proof. Suppose that $|P(t)| \leqq At^{-\alpha}$ for $t>0$, where $A>0$ and $0\leqq\alpha<1$. Then, by (1.1),

$$t|P(t)| \leqq A\mu \int_0^t (t-s)^{-\lambda} s^{-\alpha} ds ,$$

$$|P(t)| \leqq A\mu B(1-\alpha, 1-\lambda)\, t^{-\alpha-\lambda} , \qquad t>0 ,$$

where B is the beta function .[2] . Therefore,

$$P(t) = O(t^{-\alpha}) \Longrightarrow P(t) = O(t^{-\alpha-\lambda}), \qquad 0 \leqq \alpha < 1 \ . \tag{1.6}$$

By hypothesis, $P(t) = O(t^{-\alpha})$ for $\alpha = 0$. Inductively let $\alpha = 0, \lambda, 2\lambda, \ldots$ in (1.6) to prove that $P(t) = O(t^{-m\lambda})$, where m is defined by $(m-1)\lambda < 1 \leqq m\lambda$. Therefore, $P(t) = O(t^{-1+\epsilon})$ for $0 < \epsilon \leqq 1$. Another application of (1.6) yields (1.5) . So $P(t)$ is integrable and the lemma is proved.

Let $C[0,\infty)$ denote the Banach space of bounded, continuous, real functions on $[0,\infty)$ with the supremum norm. Let T denote the integral operator defined by

$$(Tf)(t) = \frac{1}{t}\int_0^t (t-s)^{-\lambda} f(s)\, ds, \qquad 0 \leqq t < \infty \ , \tag{1.7}$$

for each $f \epsilon C[0,\infty)$ such that Tf exists and $Tf \epsilon C[0,\infty)$. Then (1.1) can be expressed as the eigenvalue problem

$$TP = \frac{1}{\mu} P \ . \tag{1.8}$$

The operator T has some interesting properties not commonly encountered in the numerical analysis of integral equations. Let $f(t)$ be the characteristic function of $[\epsilon,\infty)$ with $\epsilon > 0$ in (1.7) to show that T is unbounded and, hence, not completely continuous. Since (1.1) has a non-trivial solution for every $\mu > 0$, every positive number is an eigenvalue of T .

Since T is a positive operator, it should be expected that the eigenfunction $P(t)$ is non-negative. This is also a direct consequence of the method of

successive approximations, since every approximate solution is non-negative.

Another interesting feature of the present problem is that, as we shall see, $P(t)$ and all its derivatives vanish at $t = 0$. This would seem to suggest that only the trivial solution could be obtained if the Volterra integral equation (1.1) were regarded somehow as an initial value problem. So the usual approximations methods from ordinary differential equations are not directly applicable. Nevertheless, the method of successive approximations is based to a certain extent on such an idea -- each approximate solution vanishes in some neighborhood of $t = 0$.

The existence of $P(t)$ will follow from the method of successive approximations. The uniqueness is established in Section 2 by means of Laplace transforms. The Laplace transform approach can be used also for the existence of a solution, but it is difficult to show by this means that the solution is non-negative or has other significant properties. Moreover, the method of successive approximations is applicable to more general equations (1.3) for which the Laplace transform of $P(t)$ may not be explicitly derivable.

2. Laplace Transform Analysis. Assume that $P(t)$ is a solution of (1.1) with the prescribed properties. Then the Laplace transform of $P(t)$,

$$(2.1) \qquad \hat{P}(z) = \int_0^\infty e^{-zt} P(t)\, dt \qquad (z = x + iy) ,$$

is defined for all $x \geqq 0$. Transform (1.1) and use the convolution theorem to obtain

$$(2.2) \qquad \hat{P}'(z) = -\mu \Gamma(1-\lambda) z^{\lambda-1} \hat{P}(z) , \qquad x > 0 ,$$

with the principal branch of $z^{\lambda-1}$. By (1.2),

$$(2.3) \qquad \hat{P}(0) = 1 .$$

The unique solution of (2.2) and (2.3) is

$$(2.4) \qquad \hat{P}(z) = e^{-\Omega z^{\lambda}} , \qquad \Omega = \frac{\mu\Gamma(1-\lambda)}{\lambda} ,$$

with the principal branch of z^{λ}. Since $\hat{P}(z)$ determines $P(t)$ uniquely (e.g., by means of the complex inversion integral), $P(t)$ is determined uniquely by (1.1) and (1.2).

When convenient, $P(t; \lambda, \mu)$ will be written for $P(t)$ to indicate the dependence on λ and μ. We assert that

$$(2.5) \qquad P(t; \lambda, \mu) = \left(\frac{\nu}{\mu}\right)^{1/\lambda} P\left(\left(\frac{\nu}{\mu}\right)^{1/\lambda} t ; \lambda, \nu\right) .$$

One way to prove this is to show that both members of (2.5) satisfy (1.1) and (1.2). Another way is to show that both members have the same Laplace transform,

given by (2.4) . In view of (2.5) , it suffices to determine $P(t;\lambda,\mu)$ for any single fixed value of μ .

In particular, (2.4) yields

$$\hat{P}(z) = e^{-z^{\lambda}} \quad \text{if} \quad \mu = \lambda/\Gamma(1-\lambda) \ . \tag{2.6}$$

Standard works on Laplace transforms [3], [4] give the inverse transforms of $e^{-z^{\lambda}}$ for $\lambda = \frac{1}{2}$ and $\lambda = \frac{2}{3}$. Thus,

$$P(t; \tfrac{1}{2}, \tfrac{1}{2\sqrt{\pi}}) = \frac{e^{-1/4t}}{2\sqrt{\pi}\, t^{3/2}} \tag{2.7}$$

and

$$P(t; \tfrac{2}{3}, \tfrac{2}{3\Gamma(1/3)}) = \frac{-e^{-2/27t^2}}{2\sqrt{3\pi}\, t} W_{-1/2,\,-1/6}(-4/27t^2) \ , \tag{2.8}$$

where W denotes a Whittaker function. These results and (2.5) yield $P(t; 1/2, \mu)$ and $P(t; 2/3, \mu)$ for all $\mu > 0$.

Pollard [5] derived a series expansion for the inverse transform of $e^{-z^{\lambda}}$, which yields

$$P(t; \lambda, \lambda/\Gamma(1-\lambda)) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin(n\pi\lambda)\,\Gamma(n\lambda+1)}{\pi n!\, t^{n\lambda+1}} \ . \tag{2.9}$$

Again, (2.5) yields $P(t; \lambda, \mu)$ for all λ and μ . This series is useful for calculating $P(t)$ for large values of t . However, it converges very slowly if t is only of moderate size.

Van der Corput [6] studied the asymptotic behavior as $t \to 0$ of the sum of the series in (2.9). His formulas are useful for evaluating $P(t)$ for small t. Moreover, they imply that $P(t)$ and all its derivatives vanish at $t = 0$, as was mentioned earlier. Note that (2.7) and (2.8) are consistent with this result.

Another method of calculating $P(t)$ is by means of the complex integral formula for the inverse Laplace transform. Thus, by (2.4),

$$(2.10) \qquad P(t) = \frac{1}{2\pi i} \lim_{\eta\to\infty} \int_{\xi - i\eta}^{\xi + i\eta} e^{zt} e^{-\Omega z^{\lambda}} \, dz, \qquad \xi > 0 ,$$

where

$$(2.11) \qquad z^{\lambda} = r^{\lambda} e^{i\lambda\theta} \quad \text{for } z = re^{i\theta}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} .$$

By the reflection principle,

$$(2.12) \qquad P(t) = \frac{1}{\pi} \int_{\xi + i0}^{\xi + i\infty} \mathrm{Re}\left\{ e^{zt - \Omega z^{\lambda}} \right\} dz, \qquad \xi > 0 ,$$

which is an improper Riemann integral. Let $z = \xi + iy$ to obtain

$$(2.13) \qquad P(t) = \frac{1}{\pi} \int_{0}^{\infty} e^{\xi t - \Omega r^{\lambda} \cos \lambda\theta} \cos[yt - \Omega r^{\lambda} \sin \lambda\theta] \, dy, \qquad \xi > 0 ,$$

where

$$(2.14) \qquad r = (\xi^2 + y^2)^{1/2}, \qquad \theta = \arctan[y/\xi], \qquad 0 \leqq \theta < \frac{\pi}{2} .$$

Let $\xi \to 0$ to obtain

$$(2.15) \qquad P(t) = \frac{1}{\pi} \int_0^\infty e^{-\Omega y^\lambda \cos(\lambda\pi/2)} \cos\left[yt - \Omega y^\lambda \sin(\lambda\pi/2)\right] dy \quad .$$

Although $P(t)$ can be approximated by integrating (2.13) or (2.15) numerically, a rather long calculation is involved for each value of t . The method of successive approximations presented below provides an alternative to the use of these integrals or the formulas of Pollard and Van der Corput.

3. The Function $R(t)$. The method of successive approximations will be presented first for a solution of equation (1.1) which satisfies a different normalization condition. Thus, consider the integral equation

$$(3.1) \qquad tR(t) = \mu \int_0^t (t-s)^{-\lambda} R(s)\, ds, \qquad 0 \leqq t < \infty ,$$

where $R(t)$ is bounded, continuous, non-negative, integrable, and

$$(3.2) \qquad \max_{0 \leqq t < \infty} R(t) = 1 .$$

(The maximum is attained by Lemma 1.1.) The functions $P(t)$ and $R(t)$ are related by

$$(3.3) \qquad P(t) = R(t) / \int_0^\infty R(s)\, ds , \qquad 0 \leqq t < \infty ,$$

$$(3.4) \qquad R(t) = P(t) / \max_{0 \leqq s < \infty} P(s) , \qquad 0 \leqq t < \infty .$$

The existence of both $P(t)$ and $R(t)$ will follow from the method of successive approximations. The Laplace transform analysis yields the uniqueness.

4. The Method of Successive Approximations. For $h > 0$, let $R_h(t)$, $0 \leqq t < \infty$, denote a continuous, non-negative, bounded, piecewise-linear function with possible changes in slope only at the points $t = nh$, $n = 1, 2, \ldots$. Thus, $R_h(t)$, $t \geqq 0$, is determined in terms of the values $R_h(nh)$, $n \geqq 0$, by

$$(4.1) \qquad R_h(t) = \frac{(n+1)h - t}{h} R_h(nh) + \frac{t - nh}{h} R_h((n+1)h), \quad nh \leqq t \leqq (n+1)h, \quad n \geqq 0 \; .$$

Assume that $R_h(t)$ satisfies equation (3.1) at the points $t = nh$:

$$(4.2) \qquad nh R_h(nh) = \mu \int_0^{nh} (nh - s)^{-\lambda} R_h(s)\, ds, \qquad n \geqq 0 \; .$$

Other conditions will be imposed later. Ultimately, we shall let $h \to 0$.

The procedure for deriving $R_h(t)$ may be called a trapezoidal method by analogy with the trapezoidal rule for numerical integration. Other methods, which correspond to other numerical integration rules, will be discussed in Section 7.

By (4.1) and (4.2) ,

$$nh\, R_h(nh) = \mu \sum_{k=0}^{n-1} \int_{kh}^{(k+1)h} (nh - s)^{-\lambda} R_h(s)\, ds \; ,$$

$$n R_h(nh) = \frac{\mu h^{-\lambda}}{(1-\lambda)(2-\lambda)} \left\{ \sum_{k=0}^{n-1} \left[(2-\lambda)(n-k)^{1-\lambda} - (n-k)^{2-\lambda} + (n-k-1)^{2-\lambda} \right] R_h(kh) \right.$$

$$\left. + \sum_{k=0}^{n-1} \left[(n-k)^{2-\lambda} - (n-k-1)^{2-\lambda} - (2-\lambda)(n-k-1)^{1-\lambda} \right] R_h((k+1)h) \right\} .$$

Replace k by $k-1$ in the second sum and collect terms to obtain

(4.3) $$(n-n_h)\, R_h(nh) = n_h \left[b_n\, R_h(0) + \sum_{k=0}^{n-1} c_{n-k}\, R_h(kh) \right], \qquad n \geqq 1,$$

where

(4.4) $$b_n = (2-\lambda)\, n^{1-\lambda} + n^{2-\lambda} - (n+1)^{2-\lambda}, \qquad n \geqq 1,$$

(4.5) $$c_n = (n+1)^{2-\lambda} - 2n^{2-\lambda} + (n-1)^{2-\lambda}, \qquad n \geqq 1,$$

and

(4.6) $$n_h = \frac{\mu h^{-\lambda}}{(1-\lambda)(2-\lambda)} .$$

For future reference, note that

(4.7) $$(1-n_h)\, R_h(h) = n_h (1-\lambda)\, R_h(0),$$

(4.8) $$n_h \uparrow \infty \quad \text{as} \quad h \downarrow 0,$$

and

(4.9) $$c_n > 0, \quad n \geqq 1,$$

since each c_n is a second difference of a function $t^{2-\lambda}$ with a positive second derivative.

If n_h is <u>not</u> an integer, then (4.3) determines $R_h(nh)$, $n \geqq 1$, inductively in terms of $R_h(0)$. Then (4.1) yields a solution $R_h(t)$ of (4.2). If $n_h > 1$ and $R_h(0) \gtrless 0$, then $R_h(h) \lesseqgtr 0$ by (4.7) and (4.8). Since $R_h(t) \geqq 0$ by

hypothesis, it follows that $R_h(0)$ and, hence, $R_h(t) \equiv 0$ if h is sufficiently small. This case will not be considered further.

We assume, henceforth, that n_h is a positive integer, i.e., we restrict h to the bounded and countable set

$$H = \{h: n_h = 1, 2, 3, \dots\} . \tag{4.10}$$

Now (4.3) determines $R_h(nh)$ for $1 \leqq n < n_h$, but not for $n \geqq n_h$, in terms of $R_h(0)$. As before, (4.7), (4.8) and $R_h(t) \geqq 0$ imply that $R_h(0) = 0$ if $n_h > 1$. Let $R_h(0) = 0$ for all $h \in H$. Then, by (4.3),

$$R_h(nh) = 0, \qquad 0 \leqq n < n_h . \tag{4.11}$$

Assume that

$$R_h(n_h h) > 0 . \tag{4.12}$$

Since (4.3) is now satisfied automatically for $n \leqq n_h$, it reduces to

$$(n - n_h) R_h(nh) = n_h \sum_{k=n_h}^{n-1} c_{n-k} R_h(kh), \quad n > n_h , \tag{4.13}$$

which determines $R_h(nh)$, $n > n_h$, inductively in terms of $R_h(n_h h)$. Then (4.1) yields a solution $R_h(t)$ of (4.2). Since (4.1) and (4.13) are linear relations,

$$R_h(t) = R_h(n_h h) R_h^1(t) , \tag{4.14}$$

where $R_h^1(t)$ is the particular solution with $R_h^1(n_h h) = 1$.

Equations (4.1) and (4.11) yield

$$R_h(t) = 0, \qquad 0 \leqq t \leqq (n_h - 1)h \ , \tag{4.15}$$

where $(n_h - 1)h \to 0$ as $h \to 0$ by (4.6) . Thus, both $R_h(t)$ and $R(t)$ are "very flat" at $t = 0$. By (4.1), (4.9), (4.12) and (4.13),

$$R_h(t) > 0, \qquad t > (n_h - 1)h \ . \tag{4.16}$$

Therefore, the hypothesis that $R_h(t) \geqq 0$ is satisfied.

Lemma 4.1. $R_h(t)$ is bounded and attains its maximum at some point $t = kh \leqq (\frac{\mu}{1-\lambda})^{1/\lambda}$.

Proof. The proof is based on (4.1) and (4.2) . It is analogous to that for Lemma 1.1.

Thus far, $R_h(t)$ is the general non-trivial solution of (4.2) which satisfies the given conditions. By analogy with (3.2) assume now that

$$\max_{0 \leqq t < \infty} R_h(t) = 1 \ . \tag{4.17}$$

Then $R_h(t)$ is determined completely and is given explicitly by

$$R_h(t) = R_h^1(t) / \max_{0 \leqq s < \infty} R_h^1(s) \ , \tag{4.18}$$

This formula is convenient for calculation. In view of Lemma 4.1, the maximum of $R_h^1(t)$ is a computable quantity. Each function $R_h(t)$, $h \in H$, is a continuous, piecewise-linear, non-negative approximate solution of (3.1) and (3.2).

Lemma 4.2. $R_h(t)$ is integrable on $[0,\infty)$. Moreover, for $0 < \epsilon \leqq 1$, there exists $A(\epsilon) > 0$, independent of h, such that

$$R_h(t) \leqq A(\epsilon)\, t^{-1-\lambda+\epsilon}, \qquad t > 0, \qquad h \in H \quad . \tag{4.19}$$

Proof. The proof is based on (4.1) and (4.2). It is analogous to that for Lemma 1.2.

Since $R_h(t)$ is piecewise linear and $R_h(0) = 0$,

$$\int_0^\infty R_h(t)\, dt = h \sum_{n=0}^\infty R_h(nh) \quad . \tag{4.20}$$

In view of (4.19), this quantity can be calculated to any desired accuracy.

Define

$$P_h(t) = R_h(t) \Big/ \int_0^\infty R_h(s)\, ds, \qquad t \geqq 0 \quad . \tag{4.21}$$

Then

$$nh\, P_h(nh) = \mu \int_0^{nh} (nh - s)^{-\lambda}\, P_h(s)\, ds, \qquad n \geqq 0 \quad , \tag{4.22}$$

and

(4.23) $$\int_0^\infty P_h(t)\,dt = h\sum_{n=0}^{\infty} P_h(nh) = 1 \quad .$$

Each function $P_h(t)$, $h \in H$, is a continuous, piecewise-linear non-negative approximate solution of (1.1) and (1.2) .

5. Properties of the Approximations.

Lemma 5.1. The functions $R_h(t)$, $h \in H$, are uniformly equicontinuous on each interval $t_1 \leqq t < \infty$ with $t_1 > 0$. Moreover,

$$(5.1)\qquad |R_h(t) - R_h(s)| \leqq \frac{1}{s}\left\{\frac{2^{2\lambda+1}\mu}{1-\lambda}(t-s)^{1-\lambda} + (t-s)\right\}, \quad t \geqq s > 0, \quad h \in H \quad .$$

Proof. For $n \geqq m \geqq 0$, it follows from (4.2) and (4.17) that

$$|nh R_h(nh) - mh R_h(mh)| = \mu \left| \int_0^{nh} (nh-s)^{-\lambda} R_h(s)\,ds - \int_0^{mh} (mh-s)^{-\lambda} R_h(s)\,ds \right|$$

$$\leqq \mu \left\{ \int_{mh}^{nh} (nh-s)^{-\lambda}\,ds + \int_0^{mh} [(mh-s)^{-\lambda} - (nh-s)^{-\lambda}]\,ds \right\} =$$

$$= \frac{\mu}{1-\lambda}\left\{ 2(nh-mh)^{1-\lambda} + (mh)^{1-\lambda} - (nh)^{1-\lambda} \right\} \leqq \frac{2\mu}{1-\lambda}(nh-mh)^{1-\lambda} \quad .$$

By symmetry,

$$(5.2)\qquad |nh R_h(nh) - mh R_h(mh)| \leqq \frac{2\mu}{1-\lambda} |nh - mh|^{1-\lambda} \quad , \quad m,\ n \geqq 0 \quad .$$

It follows from (4.17) and (5.2) that

$$|nh R_h(nh) - nh R_h(mh)| \leqq |nh R_h(nh) - mh R_h(mh)| + |mh R_h(mh) - nh R_h(mh)| \quad ,$$

$$(5.3)\qquad |R_h(nh) - R_h(mh)| \leqq \frac{1}{nh}\left\{ \frac{2\mu}{1-\lambda} |nh-mh|^{1-\lambda} + |nh-mh| \right\} \quad , \quad \begin{matrix} m \geqq 0 \ , \\ n > 0 \ . \end{matrix}$$

Let $nh \leqq s,\ t \leqq (n+1)h$. Then, by (4.1) and (5.3),

$$|R_h(t) - R_h(s)| = \frac{|t-s|}{h} |R_h((n+1)h) - R_h(nh)|$$

$$\leqq \frac{|t-s|}{h} \frac{1}{(n+1)h} \left\{ \frac{2\mu}{1-\lambda} h^{1-\lambda} + h \right\} = \frac{1}{(n+1)h} \left\{ \frac{2\mu}{1-\lambda} \frac{|t-s|}{h^{\lambda}} + |t-s| \right\} ,$$

so that

$$|R_h(t) - R_h(s)| \leqq \frac{1}{s} \left\{ \frac{2\mu}{1-\lambda} |t-s|^{1-\lambda} + |t-s| \right\} , \quad nh \leqq s,t \leqq (n+1)h, \ s > 0 . \tag{5.4}$$

Therefore, (5.1) is valid in this case.

Now assume that $s \leqq nh \leqq t$ for some n . Define m and n such that $(m-1)h \leqq s \leqq mh \leqq nh \leqq t \leqq (n+1)h$. By (5.3) and (5.4) ,

$$|R_h(t) - R_h(s)| \leqq |R_h(t) - R_h(nh)| + |R_h(nh) - R_h(mh)| + |R_h(mh) - R_h(s)|$$

$$\leqq \frac{1}{t} \left\{ \frac{2\mu}{1-\lambda} (t-nh)^{1-\lambda} + (t-nh) \right\} + \frac{1}{nh} \left\{ \frac{2\mu}{1-\lambda} (nh-mh)^{1-\lambda} + (nh-mh) \right\}$$

$$+ \frac{1}{s} \left\{ \frac{2\mu}{1-\lambda} (mh-s)^{1-\lambda} + (mh-s) \right\} ,$$

$$|R_h(t) - R_h(s)| \leqq \frac{1}{s} \left\{ \frac{2\mu}{1-\lambda} \left[(t-nh)^{1-\lambda} + (nh-mh)^{1-\lambda} + (mh-s)^{1-\lambda} \right] + (t-s) \right\} . \tag{5.5}$$

Since $0 < \lambda < 1$, the function $f(t) = t^{1-\lambda}$, $t \geqq 0$, is concave:

$$\frac{a^{1-\lambda} + b^{1-\lambda}}{2} \leqq \left(\frac{a+b}{2} \right)^{1-\lambda}, \quad a^{1-\lambda} + b^{1-\lambda} \leqq 2^{\lambda} (a+b)^{1-\lambda}$$

for $a, b \geqq 0$. It follows that

$$(5.6) \qquad a^{1-\lambda} + b^{1-\lambda} + c^{1-\lambda} \leqq 2^{2\lambda} (a+b+c)^{1-\lambda}, \qquad a, b, c \geqq 0 \quad .$$

Then (5.5) and (5.6) yield (5.1) for this case. Therefore, (5.1) is valid in general and the lemma is proved.

Several properties of the constants c_n defined by (4.5) will be needed in the proof of the next lemma. First, note that

$$c_n = n^{2-\lambda} \left[(1+\frac{1}{n})^{2-\lambda} - 2 + (1-\frac{1}{n})^{2-\lambda} \right] \quad .$$

Use three terms of the binomial expansions for $(1 \pm \frac{1}{n})^{2-\lambda}$ to prove that

$$(5.7) \qquad n^{\lambda} c_n \to (1-\lambda)(2-\lambda) \quad \text{as} \quad n \to \infty \quad .$$

Next, note that $c_n = c(n)$, where

$$c(t) = (t+1)^{2-\lambda} - 2t^{2-\lambda} + (t-1)^{2-\lambda}, \quad t \geqq 1 \quad .$$

Then

$$\frac{1}{2-\lambda} c'(t) = (t+1)^{1-\lambda} - 2t^{1-\lambda} + (t-1)^{1-\lambda}, \quad t \geqq 1 \quad .$$

Since this is a second difference of the function $g(t) = t^{1-\lambda}$, $t \geqq 1$, which has a negative second derivative, $c'(t) < 0$. It follows that both $c(t)$ and $\{c_n\}$ are monotone decreasing. Therefore, by (5.7) ,

$$(5.8) \qquad c_n \downarrow 0 \quad \text{as} \quad n \to \infty \quad .$$

<u>Lemma 5.2</u>. Let $0 < t_1 < \mu^{1/\lambda}$. Then, for all sufficiently small $h \in H$, the functions $R_h(t)$ are monotone non-decreasing on the interval $0 \leqq t \leqq t_1$.

<u>Proof</u>. In view of (4.11), (4.12) and Lemma 4.1, there exists a unique integer N_h for each $h \in H$ such that

$$(5.9) \qquad \begin{cases} R_h(kh) < R_h((k+1)h), & n_h - 1 \leqq k < N_h \, , \\ \\ R_h(N_h h) \geqq R_h((N_h + 1)h) \, . \end{cases}$$

Then $R_h(t)$ is non-decreasing for $0 \leqq t \leqq N_h h$. By (4.13),

$$(N_h + 1 - n_h) R_h((N_h + 1)h) - (N_h - n_h) R_h(N_h h)$$
$$= n_h \sum_{k=0}^{N_h} c_{N_h+1-k} R_h(kh) - n_h \sum_{k=0}^{N_h-1} c_{N_h-k} R_h(kh) \, .$$

Replace k by $k+1$ in the first sum and use $R_h(0) = 0$ to obtain

$$(N_h + 1 - n_h)[R_h((N_h+1)h) - R_h(N_h h)] + R_h(N_h h)$$
$$= n_h \sum_{k=0}^{N_h-1} c_{N_h-k} [R_h((k+1)h) - R_h(kh)] \, .$$

Hence, by (5.8), (5.9) and $R_h(0) = 0$,

$$R_h(N_h h) \geqq n_h c_{N_h} \sum_{k=0}^{N_h-1} [R_h((k+1)h - R_h(kh)] = n_h c_{N_h} R_h(N_h h) \, .$$

Therefore, $n_h c_{N_h} \leqq 1$. By (4.6) ,

$$c_{N_h} \leqq \frac{1}{n_h} = \frac{(1-\lambda)(2-\lambda)}{\mu} h^{\lambda} \quad , \tag{5.10}$$

so that $c_{N_h} \to 0$ as $h \to 0$. In view of (5.8), $N_h \to \infty$ as $h \to 0$. By (5.10) and (5.7) ,

$$(N_h\, h)^{\lambda} \geqq \frac{\mu (N_h)^{\lambda}\, c_{N_h}}{(1-\lambda)(2-\lambda)} \to \mu \quad \text{as} \begin{cases} h \to 0\,, \\ N_h \to \infty \end{cases} .$$

Therefore, $N_h\, h \geqq t_1$ for h sufficiently small. Since $R_h(t)$ is non-decreasing for $0 \leqq t \leqq N_h\, h$, the lemma is proved.

We conclude this section with the statement of an auxiliary result of a general nature which will be used shortly.

Auxiliary lemma. If a sequence of monotone functions converges pointwise to a continuous function on a closed interval, then the convergence is uniform.

This is an old result whose proof is not difficult. For a recent reference, see [7].

6. The Convergence of the Successive Approximations. We consider first the convergence of the functions $R_h(t)$ as $h \to 0$ through H. We shall regard $\{R_h: h \in H\}$ as a sequence ordered by letting h decrease through H or, equivalently, by letting n_h increase through the positive integers.

Lemma 6.1. Every subsequence of $\{R_h: h \in H\}$ has a further subsequence which converges uniformly on each finite interval.

Proof. Since $0 \le R_h(t) \le 1$, Lemma 5.1 implies that the Arzelà-Ascoli theorem is applicable, and by that theorem there exist successive (nested) subsequences of $\{R_h: h \in H\}$ which, respectively, converge uniformly on the intervals $[\frac{1}{m}, m]$; $m = 2, 3, 4, \ldots$. Then the usual diagonal procedure yields a single subsequence which converges uniformly on every one of the intervals $[\frac{1}{m}, m]$. Since $R_h(0) = 0$ for all $h \in H$, the subsequence converges pointwise for $t \ge 0$. It follows from Lemma 5.2 and the Auxiliary Lemma that the subsequence converges uniformly on each finite interval.

Lemma 6.2. Every subsequence of $\{R_h: h \in H\}$ which converges uniformly on each finite interval converges to a solution $R(t)$ of (3.1) with the prescribed properties.

Proof. Suppose that $R_h(t) \to R(t)$ as $h \to 0$ through a subset H' of H. The assumed uniform convergence and the continuity of each function $R_h(t)$ imply

that $R(t)$ is continuous. Since $0 \leqq R_h(t) \leqq 1$, we have $0 \leqq R(t) \leqq 1$. By (4.17) and Lemma 4.1,

$$\max_{0 \leqq t \leqq (\frac{\mu}{1-\lambda})^{1/\lambda}} R(t) = \lim_{\substack{h \to 0 \\ h \in H'}} \max_{0 \leqq t \leqq (\frac{\mu}{1-\lambda})^{1/\lambda}} R_h(t) = 1 ,$$

so that $\max_{0 \leqq t < \infty} R(t) = 1$. In (4.2), let $n \to \infty$, $h \to 0$ and $nh \to t$ with $h \in H'$ to obtain (3.1). Finally, $R(t)$ is integrable by Lemma 1.2. Thus, the lemma is proved.

<u>Theorem 6.1</u>. There exists a unique solution $R(t)$ of (3.1) with the prescribed properties, and $R_h(t) \to R(t)$ uniformly for $0 \leqq t < \infty$ as $h \to 0$ through H.

<u>Proof</u>. The existence of $R(t)$ follows from Lemmas 6.1 and 6.2. The uniqueness is a consequence of (3.4) and the uniqueness of $P(t)$, which was established in Section 2. Fix $t \in [0, \infty)$ arbitrarily and let $r(t)$ be any limit point of the numerical sequence $\{R_h(t): h \in H\}$. Then the sequence of functions $\{R_h: h \in H\}$ has a subsequence which converges at t to the value $r(t)$. By Lemmas 6.1 and 6.2, there is a further subsequence which converges to R on $[0, \infty)$. Therefore, $r(t) = R(t)$ and, hence, $R_h(t) \to R(t)$ as $h \to 0$ through H.

The pointwise convergence of $R_h(t)$ to $R(t)$ and the uniform equicontinuity established in Lemma 5.1 imply that $R_h(t) \to R(t)$ uniformly on each interval $[t_1, t_2] \subset (0,\infty)$. By Lemma 5.2 and the Auxiliary Lemma, the convergence is uniform on each finite interval. Finally, the convergence is uniform for $0 \leqq t < \infty$ by (4.19) .

Theorem 6.2. There exists a unique solution $P(t)$ of (1.1) with the prescribed properties, and $P_h(t) \to P(t)$ uniformly for $0 \leqq t < \infty$ as $h \to 0$ through H .

Proof. The existence of $P(t)$ follows from (3.3) and the existence of $R(t)$. The uniqueness was established in Section 2. By Lemma 4.2, Theorem 6.1, and the Lebesgue dominated convergence theorem

$$\int_0^\infty R_h(t)\,dt \to \int_0^\infty R(t)dt \quad \text{as} \quad h \to 0 \quad . \tag{6.1}$$

Therefore, by (3.3), (4.21) and Theorem 6.1, $P_h(t)$ converges uniformly to $P(t)$ and the theorem is proved.

In addition, it can be shown that $P_h(t) \to P(t)$ in $L_1(0,\infty)$ as $h \to 0$ through H . This is important if $P(t)$ is interpreted as a probability density.

7. Related Methods for Deriving Successive Approximations. Though our remarks in this section are, for the sake of brevity, restricted primarily to $R_h(t)$, analogous results hold for $P_h(t)$.

The trapezoidal method described in Section 4 has the following features.

(a) $R_h(t)$ is determined by linear interpolation from the values $R_h(nh)$, $n \geq 0$;

(b) $R_h(t)$ satisfies the integral equation (3.1) at the points $t = nh$, $n \geq 0$;

(c) $R_h(t)$ satisfies the normalization condition (3.2).

Variants of the method are obtained if other interpolation rules are used in (a). For example, consider $R_h(t)$ a step function such that

$$R_h(t) = R_h(nh), \quad nh \leqq t < (n+1)h, \quad n \geqq 0 . \tag{7.1}$$

It would follow from (b) that

$$nR_h(nh) = \frac{\mu h^{-\lambda}}{1-\lambda} \sum_{k=0}^{n-1} R_h(kh) \left[(n-k)^{1-\lambda} - (n-k-1)^{1-\lambda} \right], \quad n \geqq 1 . \tag{7.2}$$

Then (7.2) determines $R_h(nh)$, $n \geqq 1$, by induction in terms of $R_h(0)$, $R_h(t)$ is obtained from (7.1), and $R_h(0)$ is chosen so that (4.17) is satisfied. Theorem 6.1 is valid also for this case. Note that $R_h(0) \neq 0$, in contrast with the trapezoidal method.

As another example, consider $R_h(t)$ a step function such that

$$R_h(t) = R_h((n+1)h), \quad nh < t \leqq (n+1)h, \quad n \geqq 0 . \tag{7.3}$$

Then (b) yields, for each $n \geqq 1$,

$$(7.4)\qquad \left(n-\frac{\mu h^{-\lambda}}{1-\lambda}\right)R_h(nh) = \frac{\mu h^{-\lambda}}{1-\lambda}\sum_{k=1}^{n-1} R_h(kh)\left[(n-k+1)^{1-\lambda} - (n-k)^{1-\lambda}\right] .$$

Here, analogous to the trapezoidal method, we assume that $m_h = \frac{\mu h^{-\lambda}}{1-\lambda}$ is an integer, and let $R_h(nh) = 0$ for $0 \leqq n < m_h$. The rest of the derivation of $R_h(t)$ is similar. Again, Theorem 6.1 is valid.

A third and last example involves $R_h(t)$ determined by parabolic interpolation from three successive values $R_h(nh)$. Once again, Theorem 6.1 is valid. Further details are omitted.

Finally, it is worth noting that the original equation (1.1) provides a natural means of smoothing an approximate solution $R_h(t)$:

$$(7.5)\qquad \bar{R}_h(t) = \frac{\mu}{t}\int_0^t (t-s)^{-\lambda} R_h(s)\,ds .$$

The convergence of $\bar{R}_h(t)$ as $h \to 0$ is not investigated in this report.

8. Examples. The numerical calculation performed in support of the methods described in this paper were too extensive for complete reproduction and though a variety of values of λ and μ were considered, only two typical examples in which $\lambda = \frac{1}{2}$, $\mu = \frac{1}{2\sqrt{\pi}}$ will be discussed. Programming was performed by J. Al Abdulla and calculations were completed on the CDC 1604.

Example 1. For $\lambda = \frac{1}{2}$, $\mu = \frac{1}{2\sqrt{\pi}}$, the exact solution of (1.1) - (1.2) is given by (2.7) and (2.13). The expression in (2.13) was approximated by

$$(8.1) \qquad \frac{1}{\pi}\int_0^\beta \exp\left\{t-(1+y^2)^{1/4}\left[\frac{1+(1+y^2)^{-1/2}}{2}\right]\right\} \cdot \cos\left\{yt-(1+y^2)^{1/4}\left[\frac{1-(1+y^2)^{-1/2}}{2}\right]^{1/2}\right\} dy \quad .$$

Then, (8.1) was evaluated for a variety of values of t in the range $0 \leqq t \leqq 1$ by means of Simpson's rule with $h = .0001$. It was found that any $\beta \geqq 4500$ yielded approximations which agreed with the exact solution to at least five decimal places.

Example 2. For $\lambda = \frac{1}{2}$, $\mu = \frac{1}{2\sqrt{\pi}}$, the exact solution of (3.2) - (3.3) is

$$(8.2) \qquad R(t) = P(t)/\max_{0 \leqq s < \infty} P(s) \quad ,$$

where $P(t)$ is given by (2.7). For $n_h = 6$, h is approximately $.0039$ by (4.6), and $R_h^1(t)$ and $R_h(t)$ were generated by means of (4.13) and (4.17).

Selected results are displayed in Table I. It must be emphasized here that an unexpected stability always accompanied the trapezoidal method. In all our examples roundoff error never accumulated excessively and $R_h(t)$ always appeared to damp out even though n was taken to be as large as 100,000.

Table I

n	t = nh	$R_h^1(t)$	$R_h(t)$	$R(t)$
6	.0234	1.0000	.0012	.00195
7	.0273	4.9706	.0064	.00713
10	.0390	51.6291	.0670	.06512
14	.0546	193.8953	.2516	.24547
20	.0780	443.4639	.5755	.56781
25	.0975	599.1857	.7776	.77131
30	.1170	694.4550	.9012	.89961
38	.1482	763.3806	.9907	.98952
42	.1638	770.5151	1.0000	1.00000
45	.1755	768.5804	.9974	.99732
50	.1950	755.8736	.9810	.98282
54	.2106	740.0231	.9604	.96290
60	.2340	710.8442	.9225	.92576
62	.2418	700.2745	.9088	.91223
65	.2535	683.9963	.8877	.89135
68	.2652	667.4421	.8662	.87006
70	.2730	656.3554	.8518	.85578
78	.3042	612.5302	.7949	.79921
86	.3354	570.7857	.7407	.74516
94	.3666	531.9508	.6903	.69480
102	.3978	496.2682	.6440	.64846

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